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Optimal quadrature for analytic functions

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Abstract

Let G be a domain in the complex plane, which is symmetric with respect to the real axis and contains $[-1,1]$. For a measure τ on $[-1,1]$ satisfying a regularity condition, we determine the geometric rate of the error of integration, measured uniformly on the class of functions analytic in G and bounded by 1, if the τ-integrals are replaced by optimal interpolatory quadrature formulas with n nodes. We show that this rate is obtained for modified Gauss-quadrature formulas with respect to certain varying weights.

Keywords: Quadrature; Gauss-quadrature; Condenser; Orthogonal polynomials with varying weights

1. Introduction

Gaussian quadrature rules are a basic tool for the construction of feasible algorithms in different areas of applied mathematics. Sometimes, the classes of functions which are to be integrated bear an additional structure. In particular, the functions analytic in a given neighborhood of the interval of integration have been a class of intensive research (see, for instance, [5,6,8] and the references therein).

In a recent interesting paper, Petras [9] considers Gauss-quadrature formulas for a positive measure $d\tau(x) = w(x)\,dx$ on $[-1,1]$ of Szegő-type, i.e., for which

$$\int_0^\pi \log w(\cos t)\,dt \quad \text{exists.} \tag{1}$$

He compares these formulas with arbitrary interpolatory quadrature formulas of the form

$$Q_n[g] = \sum_{j=1}^{n} \alpha_j g(x_j), \tag{2}$$

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where $\alpha_j \in \mathbb{C}$ and $x_j \in [-1, 1]$ or, more generally, of Bakhvalov-type (see [5])

$$Q_n[g] = \sum_{j=1}^{m} \{\alpha_j \operatorname{Re} g(x_j) + \beta_j \operatorname{Re} g'(x_j) + \gamma_j \operatorname{Im} g(x_j) + \delta_j \operatorname{Im} g'(x_j)\}$$
$$+ \sum_{j=m+1}^{n} \{\alpha_j \operatorname{Re} g(x_j) + \beta_j \operatorname{Re} g(\overline{x_j}) + \gamma_j \operatorname{Im} g(x_j) + \delta_j \operatorname{Im} g(\overline{x_j})\}, \tag{3}$$

where $x_1, \ldots, x_m \in \mathbb{R}$, $x_{m+1}, \ldots, x_n \in \mathbb{C}$ and $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{C}$.

To state one of his results in a way suitable for our purposes, suppose G is a planar domain, symmetric with respect to the real axis and containing the interval $[-1, 1]$. Let τ be a finite Borel-measure on $[-1, 1]$ and denote by $\mathscr{A}_1(G)$ the class of functions analytic in G and in absolute value bounded by 1. For a quadrature formula as in (2) or (3) (with all nodes contained in G, as we will henceforth assume), define the $\mathscr{A}_1(G)$-error

$$\rho(Q_n, \mathscr{A}_1(G)) := \sup_{g \in \mathscr{A}_1(G)} \left| Q_n[g] - \int g(x)\, \mathrm{d}\tau(x) \right|. \tag{4}$$

In addition, set

$$\rho_n(\mathscr{A}_1(G)) := \inf_{Q_n} \rho(Q_n, \mathscr{A}_1(G)), \tag{5}$$

where the infimum is taken over all interpolatory quadrature formulas of the form (2) or (3). For the Gauss-quadrature formulas Q_n^g associated with a measure $\mathrm{d}\tau = w(x)\,\mathrm{d}x$ satisfying (1), Petras ([9], see also [5]) was able to show that in the special case when G is simply connected,

$$\lim_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G))} = \lim_{n\to\infty} \sqrt[2n]{\rho(Q_n^g, \mathscr{A}_1(G))} \quad (= 1/r) \tag{6}$$

if and only if G is the interior of an ellipse with foci at ± 1 (and sum of the semi-axes equal to $r > 1$).

This raises the question of how to construct optimal formulas if G is not an ellipse. For simply connected G, one may use a conformal mapping (for details, see [9]), but this approach does not work for multiply connected domains.

We want to motivate our potential theoretic approach in the following way: Relation (6) resembles the classical Bernstein theorem concerning polynomial approximation of analytic functions, for which a propagation of convergence is restricted to the largest ellipse of analyticity. Due to the interaction of approximation theoretical techniques with classical logarithmic potential theory it is possible to extend the Bernstein result to the approximation by rational functions with suitably prescribed poles [14,4]. This approach turns out to be optimal for arbitrary G, even in the sense of nth width [15]. The geometric rate of convergence can be expressed in terms of the capacity of a condenser. Keeping this in mind, it is natural to expect that modified Gaussian formulas with respect to a measure varying with a reciprocal of suitably prescribed polynomials can be good candidates for optimal formulas. Moreover, for general G, the limit in (6) should be expressable in terms of a condenser modulus.

In Section 3, we will use potential theoretic notation (which will be introduced in Section 2) in order to describe the limit of the $2n$th root of $\rho_n(\mathscr{A}_1(G))$ for quite general G and τ. In addition, we will suggest nearly optimal formulas, that can be constructed from Gaussian formulas subject to varying weights.

The proofs of these results can be found in Section 4. They are based on logarithmic potential theory and on the theory of orthogonal polynomials with respect to varying weights.

2. Notation from potential theory

Suppose μ is a (positive Borel-) measure with compact support in $\mathbb{C}$. We call

$$U(\mu,z) := \int \log \frac{1}{|z-t|}\, \mathrm{d}\mu(t) \quad (z \in \mathbb{C})$$

the logarithmic potential of μ.

Let π_k stand for the space of (complex) polynomials of degree $\leqslant k$. If $p(z) = a_k z^k + \cdots$ is a polynomial of degree k, we denote by μ_p the unit measure associating with each zero ζ of multiplicity m_ζ the equal weight m_ζ/k. For the logarithmic potential of this so-called zero counting measure of p we clearly have

$$U(\mu_p,z) = \frac{1}{k} \log \frac{|a_k|}{|p(z)|} \quad (z \in \mathbb{C}).$$

This simple, but fundamental formula shows that all information on a polynomial p is contained in the potential $U(\mu_p,\cdot)$ and its leading coefficient. In Section 4, we will often use the relation $|p(z)|/|a_k| = \exp\{-kU(\mu_p,z)\}$ to estimate the growth of the polynomial.

We will now recall the condenser problem (see [3]). Suppose E and F are disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C} \setminus (E \cup F)$ is regular with respect to the Dirichlet problem. We call the pair (E,F) a condenser. There exist unique unit measures μ_E on E and μ_F on F with minimal energy

$$I(\mu_E,\mu_F) = \int U(\mu_E,t)\, \mathrm{d}\mu_E(t) + \int U(\mu_F,t)\, \mathrm{d}\mu_F(t) - 2\int U(\mu_E,t)\, \mathrm{d}\mu_F(t)$$

subject to all choices of unit measures located on E and F, respectively. They are characterized by the fact that for certain real constants $V_E > V_F$,

$$U(\mu_E,z) - U(\mu_F,z) \begin{cases} = V_E, & z \in E, \\ = V_F, & z \in F, \\ \in [V_F, V_E], & z \in \mathbb{C}. \end{cases} \tag{7}$$

Clearly, $I(\mu_E,\mu_F) = V_E - V_F$. This quantity is called the modulus of the condenser (E,F) and denoted by $\mathrm{mod}(E,F)$. Its reciprocal $1/\mathrm{mod}(E,F)$ is known as the capacity, and the signed measure $\mu_E - \mu_F$ as the equilibrium distribution of (E,F).

The measures μ_E and μ_F are supported by the boundaries of E and F, respectively. It follows from the representation [3, p. 315] that the modulus is invariant under conformal mappings defined on $\mathbb{C} \setminus (E \cup F)$, taking into account the boundary correspondence. We also note that by means of a Möbius transform, these notions are extended to compact subsets of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ in a straightforward way (see, for details, [3]).

3. Gaussian quadrature with varying weight versus optimal quadrature

Suppose that τ is a finite measure on $[-1,1]$. To avoid trivial situations, we assume that its support consists of infinitely many points.

Let f be a continuous positive function on $[-1,1]$. We denote by $(p_n^{(f)})_{n=0}^{\infty}$ the sequence of polynomials $p_n^{(f)}(x)=\gamma_n^{(f)}x^n+\cdots\in\pi_n$, $\gamma_n^{(f)}>0$, orthonormal with respect to the measure τ/f^2 on $[-1,1]$:

$$\int_{-1}^{1}\frac{p_m^{(f)}(x)}{f(x)}\frac{p_k^{(f)}(x)}{f(x)}\,\mathrm{d}\tau(x)=\begin{cases}0 & \text{if } m\neq k,\\ 1 & \text{if } m=k.\end{cases}$$

Suppose G is a domain regular with respect to the Dirichlet problem, symmetric with respect to the real axis and containing the interval $[-1,1]$. For simplicity, we will for the moment assume that ∂G is bounded, but with appropriate modifications the following extends easily to the general case.

Let (q_{2n-1}) be a sequence of real polynomials q_{2n-1} of degree $2n-1$ or $2n-2$ with leading coefficients ±1 and the following property: The zero-counting measures $\mu_{q_{2n-1}}$ have support in a fixed bounded subset of $\mathbb{C}\setminus G$ and converge to the F-component μ_F of the equilibrium distribution of the condenser $(E,F):=([-1,1],\partial G)$ in the weak-star sense, i.e.,

$$\lim_{n\to\infty}\int h\,\mathrm{d}\mu_{2n-1}=\int h\,\mathrm{d}\mu_F\quad(h\in\mathscr{C}(\mathbb{C})).$$

For definiteness we will assume that these polynomials are positive on $[-1,1]$.

The prototype of such polynomials is as follows:

Take any array of points $y_1^{(n)},\ldots,y_n^{(n)}\in\partial G$ such that the unit measures associating with each point $y_i^{(n)}$ the mass $1/n$ converge to μ_F in the weak-star sense (e.g., rational Fekete-points on ∂G [10, p. 396], rational Fejér–Walsh-points on ∂G [12], rational Leja-points on ∂G [4, p. 99] or weighted Fekete-points on ∂G with a suitable weight [10, p. 143]) and set

$$q_{2n-1}(x):=(\pm1)\prod_{i=1}^{n-1}(x-y_i^{(n-1)})(x-\overline{y_i^{(n-1)}})\in\pi_{2n-2}. \tag{8}$$

Since $([-1,1],\partial G)$ is a symmetric condenser, which implies the symmetry of μ_F, these polynomials have the desired property.

Switching back to the general situation, denote by

$$Q_n^*[g]=\sum_{j=1}^{n}\lambda_j^{(n)}g(x_j^{(n)}) \tag{9}$$

the nth Gauss-quadrature formula associated with the measure τ/q_{2n-1}, which integrates all polynomials in π_{2n-1} exactly. Note that the $x_j^{(n)}$ are the zeros of the polynomial $p_n^{(f)}$, where $f:=(q_{2n-1})^{1/2}$. It is well known that they are all simple and contained in the open interval $]-1,1[$.

Consider the modified quadrature formula

$$\widetilde{Q}_n[g]:=\sum_{j=1}^{n}\lambda_j^{(n)}q_{2n-1}(x_j^{(n)})g(x_j^{(n)}).$$

Theorem 1. *With* $R := \exp(\mathrm{mod}([-1,1], \partial G))$, *there holds*:

$$\limsup_{n\to\infty} \sqrt[2n]{\rho(\widetilde{Q}_n, \mathscr{A}_1(G))} \leqslant \frac{1}{R}. \tag{10}$$

Remark. The proof of Theorem 1 is merely based on the possible rate of approximation of analytic functions by rational functions with prescribed scheme of poles (see [14,4] and for recent developments [1,2]). Walsh [14] uses interpolation by suitable rational interpolants and represents the remainder via the Hermite-interpolation formula. Taking a look at these representations one can find that under stronger assumptions on ∂G, e.g., if ∂G consists of finitely many smooth Jordan curves or arcs, for some constant C_0,

$$\rho(\widetilde{Q}_n, \mathscr{A}_1(G)) \leqslant C_0 \frac{1}{R^{2n}}.$$

That under certain regularity assumptions on τ the proposed quadrature formula $\widetilde{Q}_n$ is optimal in the $2n$th root sense can be seen from the lower bound in the following theorem. From now on, we drop the condition that ∂G be bounded, a condition that was only needed for a simplified introduction of the polynomials q_{2n-1}.

Theorem 2. *Suppose* $\mathrm{d}\tau = w(x)\,\mathrm{d}x$ *satisfies the Szegő-condition* (1) *and that* ∂G *consists of at most countably many components. Then*

$$\liminf_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G))} \geqslant \frac{1}{R}, \tag{11}$$

where $R := \exp(\mathrm{mod}([-1,1], \partial G))$.

In the case that G is simply connected, the assumptions on τ can be weakened. We say that $\mathrm{d}\tau = w(x)\,\mathrm{d}x$ is an Erdős-weight, if

$$w(x) > 0 \quad \text{almost everywhere on } [-1,1]. \tag{12}$$

Theorem 3. *Suppose* G *is simply connected and* $\mathrm{d}\tau = w(x)\,\mathrm{d}x$ *satisfies the Erdős-condition* (12). *Then*

$$\liminf_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G))} \geqslant \frac{1}{R}, \tag{13}$$

where $R := \exp(\mathrm{mod}([-1,1], \partial G))$.

Combining Theorems 2 and 3 with the upper bound from Theorem 1 gives:

Corollary. *Under the assumptions of Theorems* 2 *or* 3,

$$\lim_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G))} = \frac{1}{R}.$$

Remark. (i) In the case that G is simply connected, Petras [9] suggests to construct optimal quadrature formulas by transforming ordinary Gauss-quadrature formulas via the inverse of the conformal mapping that maps G onto the interior of an ellipse with foci ± 1, keeping $[-1,1]$ invariant. Only in special cases this conformal mapping is explicitly known, otherwise one has to approximate this mapping, for instance, via extremal point methods. Theorem 1 suggests to take appropriate extremal points as zeros for the denominator polynomials in order to obtain asymptotically optimal formulas.

(ii) Suppose that the zeros of the prototype polynomials q_{2n-1} in (8) do not depend on n, i.e., $y_i = y_i^{(n)} = y_i^{(m)}$ for $i \leqslant n \leqslant m$, as, for example, in the case of rational Leja-points [4, p. 99] or appropriate weighted Leja-points [10, p. 258] on ∂G. Then $q_{2n-1}(z) = (z - y_n)(z - \bar{y}_n)q_{2n-3}(z)$. Denoting by $(p_n^{(2k-1)})_{n=0}^{\infty}$ the sequence of polynomials orthonormal with respect to $\mathrm{d}w/q_{2k-1}$, we may write

$$p_n^{(2n-3)}(z)(z - y_n)(z - \bar{y}_n) = \sum_{i=0}^{n+2} d_i \, p_i^{(2n-1)}(z).$$

Since the left-hand side is orthogonal to π_{n-1} with respect to τ/q_{2n-1}, this leads to four-term recursion formulas as, for instance,

$$d_{n+2}\, p_{n+2}^{(2n-1)}(z) = p_n^{(2n-3)}(z)(z - y_n)(z - \bar{y}_n) - d_{n+1}\, p_{n+1}^{(2n-1)}(z) - d_n\, p_n^{(2n-1)}(z).$$

4. Proofs

Proof of Theorem 1. Let $g \in \mathscr{A}_1(G)$. Let $\varepsilon > 0$. By [14, Chapter VIII], the function g can be approximated by rational functions with prescribed denominator q_{2n-1} exponentially well: There exists a sequence of polynomials (p_{2n-1}^*) of degree $\leqslant 2n-1$ and a constant $M_0 = M_0(\varepsilon, G)$ such that

$$\left\| g - \frac{p_{2n-1}^*}{q_{2n-1}} \right\|_{[-1,1]} \leqslant M_0 \|g\|_G \left(\frac{1}{R} + \varepsilon\right)^{2n-1}. \tag{14}$$

Now, since Q_n^* is exact for polynomials of degree $\leqslant 2n-1$, the error of integration can be written in the form

$$\int g \, \mathrm{d}\tau - \widetilde{Q}_n[g] = \left(\int g \, \mathrm{d}\tau - \int \frac{p_{2n-1}^*}{q_{2n-1}} \, \mathrm{d}\tau \right) + (Q_n^*[p_{2n-1}^*] - \widetilde{Q}_n[g]). \tag{15}$$

For the first term in the sum on the right-hand side, we have

$$\left| \int \left(g - \frac{p_{2n-1}^*}{q_{2n-1}} \right) \mathrm{d}\tau \right| \leqslant M_0 \left(\frac{1}{R} + \varepsilon\right)^{2n-1} \tau([-1,1]). \tag{16}$$

Taking into account the polynomial exactness of Q_n^* and (14), the second term on the right-hand side of (15) can in absolute value be estimated by

$$\begin{aligned} \left| \sum_{j=1}^{n} \lambda_j^{(n)} p_{2n-1}^*(x_j^{(n)}) - \sum_{j=1}^{n} \lambda_j^{(n)} q_{2n-1}(x_j^{(n)}) g(x_j^{(n)}) \right| &\leqslant \sum_{j=1}^{n} \lambda_j^{(n)} |q_{2n-1}(x_j^{(n)})| M_0 \left(\frac{1}{R} + \varepsilon\right)^{2n-1} \\ &= M_0 \left(\frac{1}{R} + \varepsilon\right)^{2n-1} \int \frac{q_{2n-1}}{q_{2n-1}} \, \mathrm{d}\tau. \end{aligned} \tag{17}$$

Combining Eqs. (15)–(17) we obtain

$$\left| \int g \, \mathrm{d}\tau - \widetilde{Q}_n[g] \right| \leqslant 2M_0 \left(\frac{1}{R} + \varepsilon\right)^{2n-1} \tau([-1,1]).$$

Taking the supremum over all $g \in \mathscr{A}_1(G)$, taking the limit superior of the $2n$th root and letting $\varepsilon \to 0$ gives the desired estimate. □

Proof of Theorem 3. It suffices to prove the theorem for the case when G is the interior $\mathscr{E}$ of an ellipse with foci ± 1. In fact, otherwise there exists a conformal mapping Φ from the simply connected domain G onto the interior $\mathscr{E}$ of an ellipse with foci ± 1, which keeps the interval $[-1,1]$ invariant: $\Phi([-1,1]) = [-1,1]$. Then by the conformal invariance of the modulus, $\mathrm{mod}([-1,1],\partial G) = \mathrm{mod}([-1,1],\partial\mathscr{E})$. Moreover, indicating with respect to which measure the error has to be taken, $\rho_n(\mathscr{A}_1(G),\tau) = \rho_n(\mathscr{A}_1(\mathscr{E}),\tilde{\tau})$, where the transformed measure $\tilde{\tau}$, given by

$$\tilde{\tau}([a,b]) = \tau(\Phi^{-1}[a,b]) \quad (-1 \leqslant a < b \leqslant 1),$$

does also satisfy the Erdős-condition. This shows that the problem is invariant under Φ.

Now, the domain $G = \mathscr{E}$ can be written in the form of

$$G = \{z \in \mathbb{C}\colon\, g(z,\infty) < R_0\},$$

where $g(z,\infty) = \log|z + \sqrt{z^2-1}|$ is the Green's function of $\mathbb{C}\backslash[-1,1]$ with pole at infinity. The branch of the root is taken to be positive for positive z. Denote by $(p_m^{(1)})$ the sequence of polynomials orthonormal with respect to τ.

It is shown in [9, Theorem 2.1] that

$$\rho_n(\mathscr{A}_1(\mathscr{E})) \geqslant \left(\sum_{m=0}^{n} \|p_m^{(1)}\|_{\bar{G}}^2\right)^{-1}. \tag{18}$$

On the other hand, each Erdős-weight has a regular asymptotic behavior in the sense of Stahl and Totik. Therefore (see [11, Theorem 3.2.1]),

$$\limsup_{n\to\infty} \|p_n^{(1)}\|_{\bar{G}}^{1/n} \leqslant \sup_{z\in\bar{G}} \exp\{g(z,\infty)\} = \exp(R_0).$$

From this it is easy to obtain that

$$\limsup_{n\to\infty} \sqrt[n]{\sum_{m=0}^{n} \|p_m^{(1)}\|_{\bar{G}}^2} \leqslant \exp(2R_0). \tag{19}$$

The assertion follows from (18), (19) and the relation $R_0 = \mathrm{mod}([-1,1],\partial G)$. □

For the proof of Theorem 2, we need the notion of Szegő's function:

If $\mathrm{d}\tau = w(x)\,\mathrm{d}x$ is a measure on $[-1,1]$ satisfying the Szegő-condition (1), then this assumption on w guarantees the existence of the so-called Szegő-function $\mathscr{D}_w(z)$ associated with the weight w, which we may write in the form of

$$\mathscr{D}_w(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log(w(\cos\theta)|\sin\theta|)\frac{1+z\mathrm{e}^{-\mathrm{i}\theta}}{1-z\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\theta\right\} \quad (|z|<1).$$

For the basic properties of this function, see [13, Chapters X, XII].

In the remaining part of this section, $C_1, C_2, \ldots,$ and $c_1, c_2, \ldots,$ will denote positive constants possibly depending on τ and G, but not on the parameters m or n.

Proof of Theorem 2. Suppose first that the domain G is bounded and that ∂G consists of finitely many (sufficiently smooth) mutually disjoint Jordan curves.

Let $\varepsilon > 0$ be arbitrary. There exists a compact set F_ε consisting of finitely many smooth Jordan curves in $\mathbb{C}\backslash\bar{G}$, symmetric with respect to the real axis and such that

$$\mathrm{mod}([-1,1],F_\varepsilon) \leqslant \mathrm{mod}([-1,1],\partial G) + \varepsilon.$$

Since ε is arbitrary, it suffices to show that

$$\liminf_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G))} \geqslant \exp(-\mathrm{mod}([-1,1],F_\varepsilon)). \tag{20}$$

Let $\mu_1 - \mu_2$ be the equilibrium distribution of the condenser $([-1,1],F_\varepsilon)$ and denote by V_1 and V_2 the constants that $U(\mu_1,\cdot) - U(\mu_2,\cdot)$ takes on $[-1,1]$ and F_ε, respectively.

There exists a sequence (q_n) of real monic polynomials of degree n such that for $n \geqslant 1$,

$$|U(\mu_2,z) - U(\mu_{q_n},z)| \leqslant c_1 \frac{\log(n+1)}{n} \quad (z \in \bar{G})$$

(for a possible choice of p_n, see [7, Theorem 1]). Written in terms of the polynomials q_n, this means that for $z \in \bar{G}$,

$$(n+1)^{-c_1} \exp\{-nU(\mu_2,z)\} \leqslant |q_n(z)| \leqslant \exp\{-nU(\mu_2,z)\}(n+1)^{c_1}. \tag{21}$$

Consider the sequence of orthonormal polynomials $(p_m^{(q_n)})_{m=0}^{\infty}$ associated with the weight $\mathrm{d}\tau = w/(q_n)^2\,\mathrm{d}x$. Our first goal is to give estimates for the uniform norm $\|p_m^{(q_n)}/q_n\|_{\bar{G}}$. The following lemma, which will be proved at the end of this section, plays a key role.

Lemma. *There exist constants $C_0, \alpha_0, \beta_0, \delta_0 > 0$ such that for $z = \frac{1}{2}(\zeta + (1/\zeta))$ with $0 < |\zeta| < 1$, and for all $n \geqslant 1$, $n \geqslant m \geqslant 0$,*

$$|p_m^{(q_n)}(z)| \exp\{nU(\mu_1,z)\} \leqslant C_0 n^{\alpha_0} \frac{1}{|\mathscr{D}_w(\zeta)|} \frac{|\zeta|^{-\beta_0 n}}{(1-|\zeta|^2)^{\delta_0}} \exp\{nV_1\}.$$

For sufficiently small $t > 0$, $L_t := \{z\colon U(\mu_1,z) - U(\mu_2,z) = V_1 - t\}$ is a Jordan curve surrounding $[-1,1]$. Let $0 < r_2 < r_1 < 1$ be such that

$$L_t \subset \left\{ \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) : r_2 < |\zeta| < r_1 \right\}.$$

In addition, set

$$M_n(r_1,r_2) := \sup_{|\zeta| \leqslant r_1} \frac{1}{|\mathscr{D}_w(\zeta)|} \frac{r_2^{-\beta_0 n}}{(1-r_2^2)^{\delta_0}}.$$

Let $n \geqslant 1$, $0 \leqslant m \leqslant n$. The lemma implies that for $z \in L_t$,

$$|p_m^{(q_n)}(z)| \exp\{nU(\mu_1,z)\} \leqslant C_0 M_n(r_1,r_2) n^{\alpha_0} \exp\{nV_1\}.$$

By the weighted Bernstein–Walsh Lemma [10, Theorem III.2.1], this estimate can be extended so that for z on and exterior to L_t,

$$|p_m^{(q_n)}(z)| \exp\{nU(\mu_1,z)\} \leqslant C_0 M_n(r_1,r_2) n^{\alpha_0} \exp\{nV_1\}.$$

Hence, taking into account (21) and (7) for $z \in \partial G$,

$$\frac{|p_m^{(q_n)}(z)|}{|q_n(z)|} \leqslant C_1 M_n(r_1,r_2) n^{c_2} \exp\{n(-U(\mu_1,z)+V_1+U(\mu_2,z))\}$$
$$\leqslant C_1 M_n(r_1,r_2) n^{c_2} \exp\{n(V_1-V_2)\}.$$

But $p_m^{(q_n)}/q_n$ is analytic in G, so that by the maximum principle we arrive at the desired uniform estimate

$$\|p_m^{(q_n)}/q_n\|_{\bar G} = \|p_m^{(q_n)}/q_n\|_{\partial G} \leqslant C_1 M_n(r_1,r_2) n^{c_2} \exp\{n(V_1-V_2)\}. \tag{22}$$

We continue by modifying the method of Petras. Let Q_n be an arbitrary quadrature formula of type (2) or (3) with knots $y_1,\ldots,y_n$. Choose a polynomial P_n of degree n such that $P_n(y_i)=0$, $i=1,\ldots,n$, and write $P_n=\sum_{m=0}^{n} a_m p_m^{(q_n)}$. Set $P_n^* := \sum_{m=0}^{n} \bar a_m p_m^{(q_n)}$. We will now use the function

$$g := \frac{1}{\|P_nP_n^*/(q_n)^2\|_{\bar G}} \frac{P_nP_n^*}{(q_n)^2} \in \mathscr{A}_1(G)$$

to derive lower bounds for $\rho(Q_n,\mathscr{A}_1(G))$.

By the interpolating property of P_n, $Q_n[g]=0$. Therefore, taking also into account the orthogonality of the polynomials $p_m^{(q_n)}$ with respect to $\mathrm{d}\tau = w/(q_n)^2\,\mathrm{d}x$, it follows that

$$\left|Q_n[g] - \int_{-1}^{1} gw\,\mathrm{d}x\right| = \|P_nP_n^*/(q_n)^2\|_{\bar G}^{-1} \int_{-1}^{1} (P_n/q_n)^2 w\,\mathrm{d}x$$
$$\geqslant \|P_n/q_n\|_{\bar G}^{-2} \sum_{m=0}^{n} |a_m|^2. \tag{23}$$

Moreover, by Hölder's inequality and (22),

$$\|P_n/q_n\|_{\bar G}^2 \leqslant \left(\sum_{m=0}^{n} |a_m| \|p_m^{(q_n)}/q_n\|_G\right)^2$$
$$\leqslant \left(\sum_{j=0}^{n} |a_j|^2\right) (C_2 n^{c_3} M_n(r_1,r_2)^2 \exp\{2n(V_1-V_2)\}).$$

Inserting this into (23) gives

$$\rho(Q_n,\mathscr{A}_1(G)) \geqslant \left|Q_n[g] - \int gw\,\mathrm{d}x\right| \geqslant C_3 \frac{(\exp\{-(V_1-V_2)\})^{2n}}{M_n(r_1,r_2)^2 n^{c_3}}.$$

By taking the infimum over all quadrature formulas Q_n, and then the limit inferior of its $2n$th root, we obtain

$$\liminf_{n\to\infty} \rho_n(\mathscr{A}_1(G))^{1/2n} \geqslant \exp\{-(V_1-V_2)\} \liminf_{n\to\infty} (M_n(r_1,r_2))^{-1/2n}$$
$$= \exp\{-\mathrm{mod}([-1,1],F_\varepsilon)\} r_2^{\beta_0}.$$

The desired estimate (20) follows by letting $r_2 \to 1$ (thus letting $t \to 0$).

Now, suppose that G is just finitely connected. By the symmetry of G, there exists a conformal mapping Φ defined on G such that the image $\Phi(G)$ is symmetric with respect to the real axis, $\partial\Phi(G)$ consists of finitely many analytic Jordan curves and $\Phi([-1,1])=[-1,1]$. The assertion in this case follows from the first part of the proof, since the problem is invariant under conformal mappings of such type. In fact, the modulus is a conformal invariant, and Szegő-weights are transformed into Szegő-weights.

Finally, assume that ∂G is the countable union of mutually disjoint continua $F_1, F_2, \ldots$. Denote by G_k the component of $\mathbb{C}\setminus\bigcup_1^k F_j$ containing the interval $[-1,1]$, which is of course finitely connected. Then for each k,

$$\liminf_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G))} \geqslant \liminf_{n\to\infty} \sqrt[2n]{\rho_n(\mathscr{A}_1(G_k))} \geqslant \exp(-\mathrm{mod}([-1,1],\partial G_k)),$$

and by the Choquet property of the condenser capacity,

$$\mathrm{mod}([-1,1],\partial G) = \downarrow \lim_{k\to\infty} \mathrm{mod}([-1,1],\partial G_k).$$

Thus, the assertion follows by letting k tend to ∞. □

It remains to prove the lemma. In the proof we will make use of the so-called arcsine distribution, which is the unit measure μ^* on $[-1,1]$ given by

$$\mathrm{d}\mu^* = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \quad (x \in\,]-1,1[).$$

Its logarithmic potential satisfies (see [10, p. 45])

$$U(\mu^*,z) = \log 2 - \log|z+\sqrt{z^2-1}| \quad \begin{cases} = \log 2 & \text{if } z \in [-1,1], \\ \leqslant \log 2 & \text{for } z \in \mathbb{C}. \end{cases} \tag{24}$$

Proof of the lemma. We start by constructing polynomials r_n such that $|r_n/q_n|$ is in some sense nearly constant in a neighborhood of $[-1,1]$. This will allow us to apply methods from the classical Szegő theory.

It is easy to see that the measure μ_1 on $[-1,1]$ has bounded density with respect to the arcsine distribution μ^*. Consequently, there exists a number $M \in \mathbb{N}$ such that $(M+1)\mu^* - \mu_1$ is a positive measure on $[-1,1]$ of total mass M. Consider the unit measure

$$v := ((M+1)\mu^* - \mu_1)/M. \tag{25}$$

There exists a sequence of real, monic polynomials r_n of degree n such that

$$-\frac{c_4}{n}\left(\log(n+1) + \log\frac{1}{\mathrm{d}(z)}\right) \leqslant U(v,z) - U(\mu_{r_n},z) \leqslant c_4\frac{\log(n+1)}{n} \quad (z\in\mathbb{C}),$$

where $\mathrm{d}(z)$ denotes the distance from z to the interval $[-1,1]$. For instance, one may take r_n as the monic polynomial with zeros in the nth weighted Fekete points associated with the weight $U(v,\cdot)$ on $[-1,1]$ (see, for details, [7, Theorem 1]).

The above potential estimate can in terms of the polynomials r_n be written as

$$\mathrm{d}(z)^{c_4}(n+1)^{-c_4}\exp\{-nU(v,z)\} \leqslant |r_n(z)| \leqslant \exp\{-nU(v,z)\}(n+1)^{c_4}. \tag{26}$$

Note that for $x \in [-1,1]$,

$$-MU(v,x) = -(M+1)U(\mu^*,x) + U(\mu_1,x) = -(M+1)\log 2 + V_1 + U(\mu_2,x).$$

Thus, taking into account (21) and (26),

$$\|(r_m)^M/q_n\|_{[-1,1]} \leqslant (n+1)^{c_5}\exp\{n(V_1-(M+1)\log 2)\}. \tag{27}$$

From the orthonormality of the polynomials $p_m^{(q_n)}$ we have

$$1 = \int_{-1}^{1} \left(\frac{p_m^{(q_n)}(x)(r_n(x))^M}{q_n(x)(r_n(x))^M} \right)^2 w(x)\,\mathrm{d}x \geqslant \frac{1}{\|(r_n)^M q_n\|_{[-1,1]}^2} \int_{-1}^{1} (p_m^{(q_n)}(x)(r_n(x))^M)^2 w(x)\,\mathrm{d}x.$$

Applying the arguments in Szegő [13, p. 160(7.1.3–7.1.4)] to the polynomial $p_m^{(q_n)}(r_n)^M$ of degree $\leqslant m + Mn$ we find that, for $z = \frac{1}{2}(\zeta + (1/\zeta))$ with $|\zeta| < 1$,

$$|p_m^{(q_n)}(z)(r_n(z))^M \zeta^{m+Mn} \mathscr{D}_w(\zeta)|^2 \leqslant \frac{1}{\pi} \frac{1}{1-|\zeta|^2} \|(r_n)^M q_n\|_{[-1,1]}^2. \tag{28}$$

Consequently, by (26) and (27),

$$|p_m^{(q_n)}(z)| \exp\{nU(\mu_1, z)\}$$
$$\leqslant \frac{|\zeta|^{-m-Mn}}{\sqrt{\pi}\sqrt{1-|\zeta|^2}} \frac{(n+1)^{c_6}}{|\mathscr{D}_w(\zeta)|} \frac{1}{\mathrm{d}(z)^{Mc_4}} \exp\{n(V_1 - (M+1)\log 2 + U(\mu_1, z) + MU(v, z))\}.$$

The proof of the lemma is completed by inserting $\mathrm{d}(z) \geqslant (1-|\zeta|)^2$ and

$$-(M+1)\log 2 + U(\mu_1, z) + MU(v, z) = -(M+1)\log|z + \sqrt{z^2-1}| \leqslant 0$$

(see (24) and (25)) into this estimate. □

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